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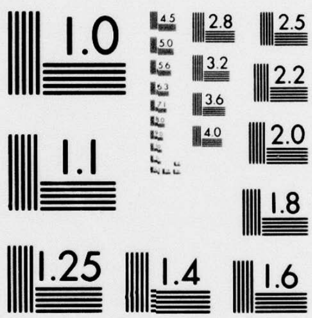
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LINEAR MULTIPLE OBJECTIVE PROBLEMS WITH INTERVAL COEFFICIENTS

by

GABRIEL R. BITRAN

Technical Report No. 167

OPERATIONS RESEARCH CENTER

MASSACHUSETTS INSTITUTE
OF
TECHNOLOGY

August 1979

SECURITY CLASSIFICATION OF THIS PAGE (*When Data Entered*)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Report No. 167	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (*and Subtitle*) 6 LINEAR MULTIPLE OBJECTIVE PROBLEMS WITH INTERVAL COEFFICIENTS.		[redacted] COVERED 9 Technical Rep[redacted]t. [redacted]
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 Gabriel R. Bitran		8. CONTRACT OR GRANT NUMBER(s) 15 N00014-75-C-0556
9. PERFORMING ORGANIZATION NAME AND ADDRESS M.I.T. Operations Research Center 77 Massachusetts Avenue Cambridge, MA 02139		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 347-027
11. CONTROLLING OFFICE NAME AND ADDRESS O.R. Branch, ONR Navy Dept. 800 North Quincy Street Arlington, VA 22217		[redacted] 11 Aug[redacted]979
		13. NUMBER OF PAGES 22 pages
14. MONITORING AGENCY NAME & ADDRESS (*if different from Controlling Office*) 14 TR-167		15. SECURITY CLASS. (*of this report*)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (*of this Report*)

Releasable without limitation on dissemination.

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

12 26p.

17. DISTRIBUTION STATEMENT (*of the abstract entered in Block 20, if different from Report*)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (*Continue on reverse side if necessary and identify by block number*)

Multiple Objective Decision Making
Vector Optimization
Efficient Points
Admissible Points

20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

See page ii.

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S/N 0102-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (*When Data Entered*)

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WITH INTERVAL COEFFICIENTS

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GABRIEL R. BITRAN

Technical Report No. 167

Work Performed Under
Contract N00014-75-C-0556, Office of Naval Research
Multilevel Logistics Organization Models
Project No. NR 347-027

Operations Research Center
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

August 1979

FOREWORD

The Operations Research Center at the Massachusetts Institute of Technology is an interdepartmental activity devoted to graduate education and research in the field of operations research. The work of the Center is supported by government contracts and grants. The work reported herein was supported by the Office of Naval Research under Contract N00014-75-C-0556.

Richard C. Larson
Jeremy F. Shapiro

Co-Directors

ABSTRACT

In this paper we consider linear multiple objective programs with coefficients of the criteria given by intervals. This class of problems is of practical interest since in many instances it is difficult to determine precisely the coefficients of the objective functions. A subproblem to test if a feasible extreme point is efficient in the problem considered is obtained. A branch and bound algorithm to solve the subproblem as well as computational results are provided. Extensions are discussed.

LINEAR MULTIPLE OBJECTIVE PROBLEMS WITH INTERVAL COEFFICIENTS

1. Introduction

The linear multiple objective problem (LMOP) consists in choosing alternatives from a polyhedral set considering simultaneously conflicting linear criteria. The LMOP is written as

$$\text{(LMOP):} \qquad \text{Max}\{Cx : x \in F\}$$

where $F = \{x \in R^n : Ax=b,\ x \geqq 0\}$, C and A are p by n and m by n matrices respectively, and $b \in R^m$. A point $x° \in F$ is said to be efficient in (LMOP), or equivalently with respect to C, if there is no $x \in F$ such that $Cx \geqq Cx°$ with at least one strict inequality. The set of efficient points is denoted by EF(C) and is considered, in this paper, to be the solution set to (LMOP).

The criteria as given by C can be seen either as the linear utilities of p decision makers or as the p objective functions of a single decision maker. Several real and potential applications of multiple criteria problems have been recently reported in areas such as water resource planning and facility location [8], scheduling of nursing resources [9], zero defects program [2], evaluation of urban policy [1], investment decision making [18], resources allocation [21], energy planning [38], macroeconomic policy [34], forest management [33], location of public facilities [28], activity planning [26], and corporate financial management [24].

Different approaches have been suggested to solve decision problems when more than one objective is considered. Survey articles by Roy [29], MacCrimmon [25], Kornbluth [23], Gal [15], Starr and Zeleny [31], Fishburn [14], Farquhar [13] and recent books by Keeney and Raiffa [22], Cohon [8], Cochrane and Zeleny [7], Zionts [37], Starr and Zeleny [30], and Hwang and Masud [19]

present an extensive coverage of the methods proposed. In this paper we concentrate on the approach that defines the set of efficient points as the solution set of the multiple criteria problem. In particular, we are interested in the set of efficient extreme points of F that we denote by $EF_{ex}(C)$. Algorithms devised by Charnes and Cooper [6], Ecker and Kouada [11], Evans and Steuer [12], Isermann [20], Gal [16], and Yu and Zeleny [36] can be used to obtain $EF_{ex}(C)$.

In practical applications it is usually difficult to determine the coefficients of the criteria matrix C because they are either specified subjectively by the decision maker(s) or they are obtained through procedures requiring subjective answers to questions posed by the analyst. In other instances, the criteria are obtained by least square minimization or by linear regression. An important example where imprecision of coefficients is known to exist and where the consideration of multiple criteria is vital, is public sector decision making. It is very difficult to estimate premiums applied to market prices to obtain social costs, shadow price of investment and rates of saving of different segments of the society ([5] and [10]). The effect of the inaccuracy in the elements of C is usually hard to evaluate and no computationally effective method is available to perform a full parametric analysis in (LMOP).

In this paper we propose the use of interval estimates for the elements c_{ij}'s of C instead of the current practice of point estimates. In most instances decision makers may feel more comfortable in specifying intervals than points; furthermore, information available from a statistical analysis for determining the criteria may be better utilized by giving the coefficients in the form of intervals. These can be seen as confidence intervals for the c_{ij}'s. In fact, situations where the proposed approach is of particular interest is when the elements of C are known to be stochastic.

The discussion above suggests the consideration of the following linear multiple objective problem with interval coefficients:

$$(P): \qquad \mathrm{Max}\{Cx : x\varepsilon F,\ C\varepsilon\Phi\}$$

where F is as previously defined and Φ is the set of p by n matrices with elements c_{ij} in the interval $[\ell_{ij},\mu_{ij}]$, $i=1,2,\ldots,p$ and $j=1,2,\ldots,n$. The lower and upper bounds ℓ_{ij} and μ_{ij} are given real numbers. Problem (P) represents a family of (LMOP)'s, with one problem for each C in Φ. The solution set to (P) is defined as the set of points in F efficient with respect to every $C\varepsilon\Phi$ and is denoted by EF. We also refer to it as the set of efficient points in (P). The definition of EF implies that $EF = \bigcap_{C\varepsilon\Phi} EF(C)$.

The plan of the paper is as follows. In section 2 properties of (P), including connectedness of EF and the existence of efficient points, are discussed. In section 3 the case where all lower bounds ℓ_{ij} are nonnegative is considered. An algorithm to determine EF_{ex} is obtained in section 4. It is shown that to determine the set of efficient extreme points in (P) it suffices to use any of the algorithms to solve (LMOP), for a fixed C, in conjunction with a subroutine that solves a test problem at each extreme point. Although the test problem is nonlinear, an effective implicit enumeration scheme that only considers linear programs has been developed to solve it. Computational results, extensions, and topics for further research are discussed in section 5. Straightforward proofs are omitted for the benefit of compactness of the paper.

The following notation is given for future references. Lower case letters are used to denote vectors. Superscripts differentiate vectors and subscripts indicate the components of a vector. The partial ordering relation $x \geq y$ means $x_j \geqq y_j$, for all j, with at least one strict

inequality. Matrices are denoted by capital letters. The i^{th} row and j^{th} column of a matrix C are written as $C_{i.}$ and $C_{j.}$, respectively. Given two matrices $C^1 = [c^1_{ij}]$, $C^2 = [c^2_{ij}]$ the notation $C^1 \leqq C \leqq C^2$ indicates that the matrix $C = [c_{ij}]$ has each element c_{ij} satisfying $c^1_{ij} \leqq c_{ij} \leqq c^2_{ij}$. Given a set S, we denote the subset of its extreme points by S_{ex}. Max{....} (max{...}) represents a multiple (a single) objective maximization problem. Consider the problem $\text{Max}\{Cx : Ax=b,\ x \geqq 0\}$ and let $x°$ be an extreme point of the constraint set. Without loss of generality the basis B, corresponding to $x°$, is assumed to be composed by the first columns of A, $x°$ is written as $x° = (x^{\circ B}, x^{\circ N})$, A and C are accordingly partitioned as $A = (B, A^N)$ and $C = (C_B, C_N)$ and the linear system multiplied by B^{-1} becomes $I_m x^{\circ B} + \bar{A}^N x^{\circ N} = \bar{b}$ where $\bar{A}^N = B^{-1}A^N$ and $\bar{b} = B^{-1}b$.

2. Properties of (P)

The set Φ is contained in the vector space of p by n matrices and is characterized as follows.

Proposition 2.1: Φ is a convex set with extreme points having each element c_{ij} at its upper or lower bound.

For each $C \varepsilon \Phi$ let $K(C) = \{p \varepsilon R^n : Cp \geq 0\}$ and let $K(\Phi) = \bigcup_{C \varepsilon \Phi} K(C)$. K(C) and $K(\Phi)$ are called the preference cones associated with matrix C and problem (P), respectively. Note that the elements of K(C) are the directions of preference, i.e., if $y = p+x$ for some $p \varepsilon P(C)$ then $Cy \geq Cx$. Preference cones play an important role in general multiple criteria optimization (see [35] and [4]). They provide an insightful geometrical interpretation and are intimately related to the connectedness and existence of efficient points. Denote by M the subset of matrices of Φ having all elements of each column at the upper bound or at

the lower bound. Hence, if $C \epsilon M$, for $j=1,2,\ldots,n$ either $C_{\cdot j} = U_{\cdot j}$ or $C_{\cdot j} = L_{\cdot j}$ where $U = [\mu_{ij}]$ and $L = [\ell_{ij}]$. The maximum number of elements in M is 2^n.

<u>Proposition 2.2</u>: $K(\Phi) = \bigcup_{C \epsilon M} K(C) \triangleq K(M)$.

Proof: Assume $p \epsilon K(\Phi)$. Thus, there is $C \epsilon \Phi$ such that $Cp \geq 0$. Consider the matrix $C^1 = [c^1_{ij}]$ defined for $j=1,2,\ldots,n$ as $C^1_{\cdot j} = L_{\cdot j}$ if $p_j < 0$ and $C^1_{\cdot j} = U_{\cdot j}$ otherwise. Then, $C^1p \geqq Cp \geq 0$ and $p \epsilon K(C^1)$. Since by definition $C^1 \epsilon M$ it follows that $K(\Phi) \subseteq K(M)$. The reverse inclusion relation is a direct consequence of the definitions of $K(\Phi)$ and $K(M)$. ▨

Proposition 2.2 shows that from the infinite family of problems represented by (P) we just have to consider the finite family of (LMOP)'s having C in M.

<u>Corollary 2.3</u>: $EF = \bigcap_{C \epsilon M} EF(C) \triangleq EF(M)$.

Proof: By definition $EF = \bigcap_{C \epsilon \Phi} EF(C)$. Since $M \subseteq \Phi$, $EF \subseteq EF(M)$. Assume $x \epsilon F$ and $x \notin EF$. Thus, $[\{x\} + K(\Phi)] \cap F \neq \{x\}$. By Proposition 2.2 it follows that $[\{x\} + K(M)] \cap F \neq \{x\}$ and therefore $x \notin EF(C)$ for some $C \epsilon M$. Consequently $EF(M) \subseteq EF$ and hence $EF = EF(M)$. ▨

Proposition 2.2 and Corollary 2.3 imply that determining the efficient points in (P) is equivalent to solve

$$(P'): \qquad \text{Max}\{Cx : x \epsilon F,\ C \epsilon M\}.$$

This last problem represents a finite but large family of (LMOP)'s. The algorithm developed in section 4 solves (P') using an efficient implicit enumeration scheme.

The linear multiple objective problem, LMOP, has been extensively studied, (e.g., [12],[27],[16],[36],[39], and [11]). It is well known that EF(C) is closed and connected, the existence of an efficient point implies the existence of an efficient extreme point, the preference cone K(C) is convex

and whenever a point in the relative interior of a face is efficient the whole face also is. The following proposition establishes the corresponding results in problem (P).

Proposition 2.4: a) EF is closed,

b) If a point in the *relative interior of a face of F is* efficient, the whole face also is,

c) If EF $\neq \phi$, there is an extreme point of F efficient in (P),

d) $K(\Phi)$ is not necessarily convex and,

e) EF is not necessarily connected.

Proof: a), b), and c) follow trivially from the properties of (LMOP) and Corollary 2.3. d) and e) are illustrated with examples. Let Φ be in the space of 1 by 2 matrices and let $\mu_{11} = 1$, $\ell_{11} = .5$, $\mu_{12} = 1$, and $\ell_{12} = .5$. Consider $p^1 = (.6,-1)$, $C^1 = [1,.5]\varepsilon\Phi$, $p^2 = (-1.8,1)$, and $C^2 = [.5,1]\varepsilon\Phi$. Then, $C^1p^1 > 0$, $C^2p^2 > 0$ but $p = p^1+p^2 = (-1.2,0)\notin K(\Phi)$. It is not difficult *to construct examples where EF is not connected when the closure of* $K(\Phi)$ contains a non-trivial subspace as is shown, for general multiple objective optimization problems, in [4]. However in (P), even when this is not the case, EF may be not connected. For example, consider the case where Φ is in the space of 4 by 3 matrices and $\ell_{11} = 1$, $\mu_{11} = 2$, $\ell_{12} = \mu_{12} = \ell_{13} = \mu_{13} = \ell_{23} = \mu_{23} = \ell_{32} = \mu_{32} = 1$, $\ell_{43} = \mu_{43} = -1$, $\ell_{21} = \mu_{21} = \ell_{22} = \mu_{22} = 2$, and $\ell_{31} = \mu_{31} = \ell_{33} = \mu_{33} = \ell_{41} = \mu_{41} = \ell_{42} = \mu_{42} = 0$. Hence the subset M of Φ is composed by the two matrices

$$C^1 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and } C^2 = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Let F be defined as the convex hull of A, B, C, and D where $A = (0,0,0)$, $B = (3,-3,0)$, $C = (-\frac{1}{3}, 0, \frac{2}{3})$, and $D = (5, -\frac{11}{2}, 1)$. Point D is in the plane defined by A, B, and C. It can be shown that no point in the relative interior of ABCD is in $EF(C^1)$ or in $EF(C^2)$, $EF(C^1) = AC \cup CD$ and $EF(C^2) = AB \cup BD$. Therefore $EF = EF(C^1) \cap EF(C^2) = \{A,D\}$, i.e., EF has two elements, the points A and D. Moreover, the vertices A and D are not adjacent in F. ▨

Consider problem (LMOP). It is well known [17] that $x^\circ \varepsilon EF(C)$ if and only if x° solves $\max\{\lambda Cx : x \varepsilon F\}$ for some $\lambda > 0$, $\lambda \varepsilon R^p$. It is therefore natural to consider systems (I) and (II), below, in order to obtain conditions for EF to be non-empty.

$$\begin{cases} \lambda^i C^i - qI_n = 0 & \text{for all } i \text{ such that } C^i \varepsilon M \\ \lambda^i > 0 & \\ \lambda^i \varepsilon R^p, \; q \varepsilon R^n & \end{cases} \qquad \text{(I)}$$

and

$$\begin{cases} \lambda^i C^i - qI_n = 0 & \text{for all } i \text{ such that } C^i \varepsilon M \\ \lambda^i \geqq 0 & \\ \lambda^i \varepsilon R^p, \; q \varepsilon R^n & \end{cases} \qquad \text{(II)}$$

Clearly, if (I) has a solution q° and if the problem (Pq°): $\max\{q^\circ x : x \varepsilon F\}$ has an optimal solution then $EF \neq \phi$. When F is bounded the condition on (Pq°) is always satisfied and can be deleted. Also, if for some $C \varepsilon M$ (or Φ), $K(C) \cap \{r \varepsilon R^n : Ar=0, r \geqq 0\} \neq \phi$, EF has no elements since $EF(C) = \phi$. Whenever system I has no solution it follows from Proposition A.1 in [4] that the closure of the convex hull of $K(\Phi)$ contains a non-trivial subspace. If in addition, system II has the unique solution $q^\circ = 0$ then, the subspace in the closure of the convex hull of $K(\Phi)$ is R^n. When $q^\circ \neq 0$ is feasible in system II and the problem (Pq°) : $\max\{q^\circ x : x \varepsilon F\}$ has a unique optimal solution x°, EF is

non-empty since $x^\circ \varepsilon EF(C)$ for all $C \varepsilon M$. The two following examples illustrate the necessity of the condition of (Pq°) to have a unique solution and that EF need not be empty when the subspace in the closure of the convex hull of $K(\Phi)$ is R^n. Let C^1 and C^2 be the only two elements of M where,

$$C^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } C^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and let F be the segment AB where A = (-1,0) and B = (1,0). We have that $A \varepsilon EF(C^2)$, $B \varepsilon EF(C^1)$, and $EF(M) = \phi$. Note that in this case system I has no solution, system II has solution $q^\circ = (0,1)$ and the segment AB is optimal in (Pq°). If instead of the segment AB, F is defined as the triangle ABD with D = (0,1) then, D becomes the unique solution of (Pq°) and $D \varepsilon EF$. The next example illustrates the situation where $F \subseteq R^2$, closure of the convex hull of $K(\Phi)$ is R^2 but $EF \neq \phi$. Let $C^1 = [1,0]$ and $C^2 = [-1,0]$ be the only two elements in M and let F be the line defined by the two points A = (0,1) and B = (0,-1). Thus, $K(C^1) = \{p \varepsilon R^2 : p_1 > 0\}$, $K(C^2) = \{p \varepsilon R^2 : p_1 < 0\}$; closure of the convex hull of $K(\Phi)$ is R^2, but all points in F are in $EF(C^1)$ and $EF(C^2)$ and therefore F = EF.

3. Nonnegative Criteria

In this section we concentrate on the case where $\ell_{ij} \geqq 0$ for all (i,j). Without loss of generality we assume that for each $j = \{1,2,\ldots,n\}$ there is a C in Φ with the elements in column n not all zeros. The next two propositions suggest a simple algorithm to determine EF_{ex} when the criteria are nonnegative.

Proposition 3.1: Let $EF(I_n)$ be the set of efficient points in problem $P(I_n) : \text{Max}\{I_n x : x \varepsilon F\}$. Then, $EF \subseteq EF(I_n)$.

<u>Proposition 3.2</u>: Assume $x^\circ \varepsilon EF(I_n)$ and $x^\circ \notin EF$. Then, there is $y^\circ \varepsilon EF(I_n)$ such that $Cy^\circ \geq Cx^\circ$ for some C in Φ.

Proof: $x^\circ \notin EF$ implies that there is a matrix $C\varepsilon\Phi$ such that $x^\circ \notin EF(C)$. Let $e=(1,1,\ldots,1)\varepsilon R^p$ and consider the problem

$$(\bar{P}): \quad \max\{eCx : Cx \geqq Cx^\circ,\ Ax=b,\ x \geqq 0\}.$$

If $(\bar{P})$ is unbounded there is $r\varepsilon R^n$ satisfying $Cr \geqq 0$, $Ar=0$, $r \geqq 0$, $eCr>0$ consequently, $Cr \geq 0$ and $r \geq 0$. Thus, $r+x^\circ \varepsilon F$ and $r+x^\circ \geq x^\circ$ contradicting the assumption $x^\circ \varepsilon EF(I_n)$. Therefore, $(\bar{P})$ is bounded. Furthermore y° solves $(\bar{P})$ implies that $y^\circ \varepsilon EF(I_n)$ since otherwise, there would be a $z\varepsilon F$, $z \geq y^\circ$, $Cz \geq Cy^\circ$ and $eCz>eCy^\circ$. Hence, any y° optimal in $(\bar{P})$ is in $EF(I_n)$ and satisfies $Cy^\circ \geq Cx^\circ$.

Propositions 3.1 and 3.2 show that EF is contained in the set $EF(I_n)$ and for each x° efficient in $P(I_n)$ and not efficient in (P) there is an element $y^\circ \varepsilon EF(I_n)$ and a matrix C in Φ that can be used to eliminate x° from $EF(I_n)$, i.e., such that $Cy^\circ \geq Cx^\circ$. The two propositions suggest the following algorithm to obtain the set of efficient extreme points in (P).

<u>Algorithm</u>

Step 1: Using any algorithm for linear multiple objective problems, determine $EF_{ex}(I_n)$.

Step 2: For each $x^\circ \varepsilon EF_{ex}(I_n)$ and each $x\varepsilon EF_{ex}(I_n)$, $x \neq x^\circ$, define the matrix $\tilde{C} = [\tilde{c}_{ij}]$ as follows:

$$\tilde{C}_{\cdot j} = U_{\cdot j} \quad \text{if } x_j - x^\circ_j \geqq 0 \text{ and}$$
$$\tilde{C}_{\cdot j} = L_{\cdot j} \quad \text{if } x_j - x^\circ_j < 0 \text{ for } j=1,2,\ldots,n.$$

Note that $\tilde{C}\varepsilon M$. If $\tilde{C}(x-x^\circ) \geq 0$, $x^\circ \notin EF_{ex}$ and x° can be discarded from $EF_{ex}(I_n)$. Otherwise, choose another x from the set of efficient points in $P(I_n)$ and repeat the procedure. If x° is not discarded after being compared with the elements of $EF_{ex}(I_n)$ it follows from Proposition 3.2 that it is efficient in

(P). Proposition 3.1 ensures that all elements of EF_{ex} can be obtained by this algorithm. It is interesting to note that Proposition 3.1 is valid whenever the criteria are non-negative. Therefore if for sensitivity analysis purposes we change the upper or lower bound on any c_{ij} we can restart the problem with the same $EF_{ex}(I_n)$ as long as $\ell_{ij} \geq 0$ for all (i,j) and the columns with all components equal to zero in all matrices in Φ remain unchanged. Related results were developed by the author [3] for the zero-one linear multiple objective problem with nonnegative criteria.

4. The General Linear Case

Basically, the existing algorithms to determine the efficient extreme points of (LMOP) move from one element of $EF_{ex}(C)$ to an adjacent one. The new extreme point is chosen by a test rule. The solution procedure, that we propose, to obtain EF_{ex} differs from the one described in that we solve an additional subproblem to check if the extreme point is in EF_{ex}. To simplify the notation we assume that all extreme points of F, under consideration, correspond to non-degenerate primal basis. At the end of the section it is mentioned how this assumption can be easily relaxed.

Initially we consider the problem of how to determine if a given extreme point of F is or is not efficient in (P).

Proposition 4.1: Let $x^\circ = (x^{\circ B}, x^{\circ N})$ be a non-degenerate extreme point of F. Then, $x^\circ \varepsilon EF_{ex}$ if and only if the system

$$\left.\begin{array}{l} (C_N - C_B\bar{A}^N)\mu \geq 0 \\ \mu \geq 0, \quad C = (C_B, C_N)\varepsilon M \end{array}\right\} \qquad (4.1)$$

has no solution.

Proof: Follows from the definitions of EF, M, and lemma 2.1 in [12].

Proposition 4.1 can be written in an operational form as: "$x^\circ \varepsilon EF_{ex}$ if and only if the optimal value of $P(x^\circ)$: $z=\max\{es: (C_B\bar{A}^N - C_N)\mu + I_p s = 0, \mu \geqq 0, s \geqq 0, C\varepsilon M\}$ is zero where, $e = (1,1,\ldots,1)\varepsilon R^p$". The best choice of $C = (C_B, C_N)$ is such that $C_N = U_N$ where $U = (U_B, U_N)$, i.e., every element of C_N should be chosen at its upper limit. Hence,

<u>Corollary 4.2</u>: There is always an optimal solution (C^*, μ^*, s^*) of $P(x^\circ)$ with $C_N^* = U_N$.

The reader should note that $P(x^\circ)$ is not a linear programming problem since besides μ and s, matrix C_B is also unknown. C_B is p by m if A has full row rank, and does not depend on n. The next propositions play an important role in the algorithm developed to solve $P(x^\circ)$.

Let N denote the set of indices corresponding to the non-basic components of x°. For each $j\varepsilon N$ let $C_B(j)$ be the p by m matrix with columns $C_B(j)_{\cdot k}$ defined as follows:

$$C_B(j)_{\cdot k} = \begin{cases} = (L_B)_{\cdot k} & \text{if } \bar{A}^N_{kj} \geqq 0 \text{ and} \\ = (U_B)_{\cdot k} & \text{if } \bar{A}^N_{kj} < 0 \qquad k=1,2,\ldots,m \end{cases}$$

where $L = (L_B, L_N)$ and $U = (U_B, U_N)$. We refer to $C_B(j)$ as the <u>ideal matrix</u> corresponding to index j. It is the p by m matrix in (L_B, U_B) that when multiplied by $\bar{A}^N_{\cdot j}$ gives the p vector with the lowest component values. Hence,

<u>Proposition 4.3</u>: a) $C_B(j)_{\cdot k}\, \bar{A}^N_{kj} \leqq (C_B)_{\cdot k}\, \bar{A}^N_{kj}$ $j\varepsilon N$; $k=1,2,\ldots,m$ and $L_B \leqq C_B \leqq U_B$,

b) $C_B(j)\, \bar{A}^N_{\cdot j} \leqq C_B \bar{A}^N_{\cdot j}$ for all $j\varepsilon N$ and $L_B \leqq C_B \leqq U_B$.

<u>Corollary 4.4</u>: If $P(x^\circ)$ has a feasible solution $C^* = (C_B^*, U_N)$, s^*, μ^* with $es^* > 0$ then $P(x^\circ, m_1)$ defined as

$P(x^\circ, m_1)$: $v = \max es$

$$\text{s.t.} \quad \sum_{j\varepsilon N} \left\{ \left[\sum_{i=1}^{m_1} (C_B)_{\cdot i}\, \bar{A}^N_{ij} + \sum_{k=m_1+1}^{m} C_B(j)_{\cdot k}\, \bar{A}^N_{kj} \right] - (U_N)_{\cdot j} \right\} \mu_j + I_p s = 0$$

$$\mu \geqq 0,\ s \geqq 0,\ (LB)_{\cdot i} \leqq (C_B)_{\cdot i} \leqq (U_B)_{\cdot i} \qquad i=1,\ldots,m_1.$$

has a feasible solution with $es > 0$ for all $0 \leqq m_1 \leqq m$ [for $m_1 = 0$ ($m_1=m$) the first (second) summation vanishes and $P(x^\circ, m_1=m) = P(x^\circ)$].

Proof: By Proposition 4.3a) $C_B(j)_{\cdot k}\, \bar{A}^N_{kj} \leqq (C_B)_{\cdot k}\, \bar{A}^N_{kj}$. Hence,

$$s = -\sum_{j\varepsilon N} \left\{ \sum_{i=1}^{m_1} (C_B^*)_{\cdot i}\, \bar{A}^N_{ij} + \sum_{k=m_1+1}^{m} C_B(j)_{\cdot k}\, \bar{A}^N_{kj} - (U_N)_{\cdot j} \right\} \mu_j^* \geqq$$

$$\geqq -\sum_{j\varepsilon N} \left\{ \sum_{i=1}^{m} (C_B^*)_{\cdot i}\, \bar{A}^N_{ij} - (U_N)_{\cdot j} \right\} \mu_j^* = s^*.$$

Therefore $es \geqq es^* > 0$.

If x° is not efficient in (P) it follows from Proposition 2.2 and Corollary 4.2 that there is a matrix $C = (C_B, C_N=U_N)\varepsilon M$ and an $x\varepsilon F$ such that $C(x-x^\circ) \geq 0$. Thus, a naive method is to solve $P(x^\circ)$ for all matrices C_B with each column having all elements at the upper or lower bound. Since there can be 2^m of such matrices this procedure would require a significant amount of computer time. Instead, we perform the following implicit enumeration which considers a sequence of linear programs in order to solve the non-linear problem $P(x^\circ)$.

Implicit Enumeration Algorithm

Description: Start by solving $P(x^\circ,\ m_1=0)$. If $v=0$ stop, x° is efficient in (P). If $v > 0$ let $m_1=1$ and generate the following two problems

$P(x^\circ, m_1=1,1)$: max es

$$\text{s.t.} \sum_{j\in N} \left\{[(U_B)_{\cdot 1} \bar{A}^N_{1j} + \sum_{k=2}^{m} C_B(j)_{\cdot k} \bar{A}^N_{kj}] - (U_N)_{\cdot j}\right\} \mu_j + I_p s = 0$$

$$\mu \geqq 0, \ s \geqq 0$$

and

$P(x^\circ, m_1=1,0)$: max es

$$\text{s.t.} \sum_{j\in N} \left\{[(L_B)_{\cdot 1} \bar{A}^N_{1j} + \sum_{k=2}^{m} C_B(j)_{\cdot k} \bar{A}^N_{kj}] - (U_N)_{\cdot j}\right\} \mu_j + I_p s = 0$$

$$\mu \geqq 0, \ s \geqq 0.$$

Where in the notation $P(x^\circ, m_1=1,t)$, $t=1$ $(t=0)$ indicates that the column, in C_B, corresponding to $m_1=1$ has all its elements at the upper (lower) bound. If the optimal value of $P(x^\circ, m_1=1,1)$ is zero, by Corollary 4.4, there is no optimal matrix C_B in $P(x^\circ)$, with $z > 0$ and having the first column equal to $(U_B)_{\cdot 1}$. In this case we do not need to consider any descendent of $P(x^\circ, m_1=1,1)$ and the branch is fathomed. If the optimal value of $P(x^\circ, m_1=1,1)$ is positive we generate two new problems $P(x^\circ, m_1=2,1,1)$ and $P(x^\circ, m_1=2,1,0)$. These two problems are obtained by substituting $C_B(j)_{\cdot 2}$, in $P(x^\circ, m_1=1,1)$, respectively by $(U_B)_{\cdot 2}$ and $(L_B)_{\cdot 2}$. We proceed in the same way, i.e., branching on problems with optimal value positive and fathoming those with optimal value zero until, we either conclude that x° is efficient in (P) or obtain a C_B such that $L_B \leqq C_B \leqq U_B$ and $z > 0$. An example of a tree generated by the implicit enumeration algorithm is given in Figure 1. In that figure, $P(x^\circ, m_1=3,1,0,1)$ is the problem

max es

$$\text{s.t.} \sum_{j\in N} \left\{[U_{\cdot 1} \bar{A}^N_{1j} + L_{\cdot 2} \bar{A}^N_{2j} + U_{\cdot 3} \bar{A}^N_{3j} + \sum_{k=4}^{m} C_B(j)_{\cdot k} \bar{A}^N_{kj}] - (U_N)_{\cdot j}\right\} \mu_j + I_p s =$$

$$\mu \geqq 0, \ s \geqq 0.$$

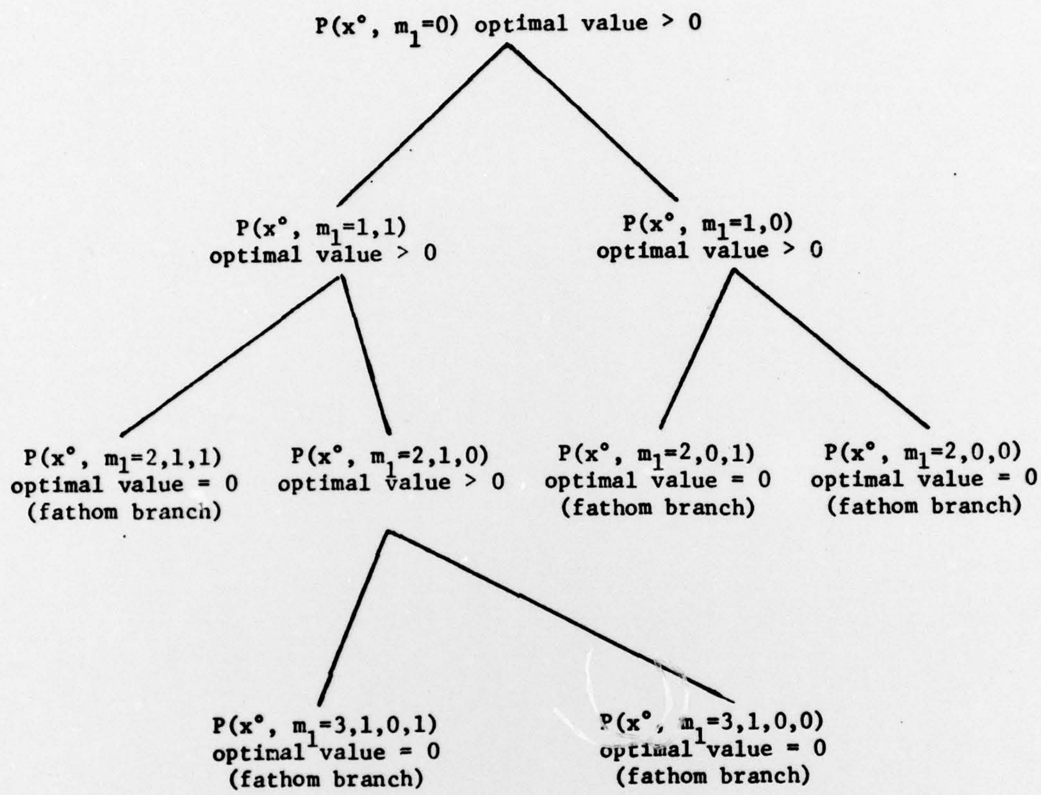

Figure 1: Example of a Tree Generated by the Implicit Enumeration Algorithm. In this case x° is efficient in (P).

The convergence of the algorithm, after solving a finite number of linear programs, follows from Corollary 4.4 and the fact that the number of matrices C_B that can possibly be enumerated is finite.

In (LMOP) the set of efficient points is connected and the algorithms to determine $EF_{ex}(C)$ move from one efficient extreme point to another by using only efficient edges. In (P), the set of efficient points is not necessarily connected, as illustrated in section 2. However, whenever it is of interest to identify the connected subsets of EF the following problem can be solved to determine if an adjacent edge to the efficient point $x°$ is efficient or not:

$$\bar{Q}(x°,j): \quad \max\ es$$

$$\text{s.t.} \quad (C_B\bar{A}^N - C_N)\mu - [C_B\bar{A}^N_{\cdot j} - (C_N)_{\cdot j}]w + I_p s = 0$$

$$\mu \geqq 0, \ s \geqq 0$$

where x_j is a non-basic variable, in the feasible solution associated with $x°$ which when introduced into the basis generates an adjacent extreme point that we denote by x^j. The edge of F defined by the pair $(x°,x^j)$ is the segment $r^j = [x°,x^j]$.

Proposition 4.5: Let $x°$ be an efficient extreme point in F. Then, the edge $r^j = [x°,x^j]$ is efficient in (P) if and only if the optimal value of $\bar{Q}(x°,j)$ is zero.

Proof: Follows by the definitions of EF, M, and lemma 2.3 in [12]. ▨

Corrolary 4.6: Let $x°$ be an efficient extreme point in F. Then, the edge $r^j = [x°,x^j]$ is efficient in (P) if and only if the optimal value of the problem problem

$Q(x^\circ,j)$: max es

$$\text{s.t.}\quad (C_B\bar{A}^N - U_N)\mu - [\bar{C}_B(j)\bar{A}^N_{\cdot j} - (L_N)_{\cdot j}]w^+ + [C_B(j)\bar{A}^N_{\cdot j} - (U_N)_{\cdot j}]w^- + I_p s = 0$$

$$\mu \geqq 0,\ w^+ \geqq 0,\ w^- \geqq 0,\ s \geqq 0$$

is zero where, $\bar{C}_B(j)$ is the p by m matrix with columns $C_B(j)_{\cdot k}$ defined as

$$\bar{C}_B(j)_{\cdot k} = \begin{cases} = (U_B)_{\cdot k} & \text{if } \bar{A}^N_{kj} \geqq 0 \text{ and} \\ = (L_B)_{\cdot k} & \text{if } \bar{A}^N_{kj} < 0 \end{cases}$$

Proof: Follows from the fact that the optimal values of $Q(x^\circ,j)$ and $\bar{Q}(x^\circ,j)$ are equal.

Clearly, $Q(x^\circ,j)$ can also be solved by the implicit enumeration algorithm. To obtain EF_{ex} we can choose a matrix C, for example $C = \frac{1}{2}(L+U)$, apply any of the existing algorithms to solve (LMOP) and at each of its efficient extreme points x° solve $P(x^\circ)$ to determine if it is or not in EF_{ex}. This procedure will generate the whole set EF_{ex} since it is contained in $EF_{ex}(C)$. Therefore, the method consists in using the connected graph formed by efficient extreme points and edges of (LMOP) to determine the corresponding non-connected subgraph of (P). It is worthwhile to point out that the algorithms developed by Yu and Zeleny [36], Ecker and Kouada [11], and Gal [16] to obtain efficient faces of F can also be conveniently adapted to determine the efficient faces of F in problem (P). Similarly the implicit enumeration algorithm can be applied in conjunction with Steuer's interval weights method [32]. To conclude this section, we consider the non-degeneracy assumption made earlier. Whenever x° is a degenerate basic feasible solution it is necessary to add to problems $P(x^\circ)$ and $P(x^\circ,m_1)$ the constraints $\bar{A}^N_{i\cdot}\mu \leqq 0$ for all i such that the corresponding basic components of x° have value zero.

5. Computational Results and Conclusions

The implicit enumeration algorithm was tested by solving four hundred problems of the form of $P(x°)$. The matrices A were randomly generated in the interval [0,20] with 20% of negative elements and the value of the right-hand-side vector b was fixed with all components equal to 100. The criteria matrices were constructed as follows. For each problem a matrix $C = [c_{ij}]$ was randomly generated in the interval [0,99] and the matrices U and L were defined as $U = \beta C$ and $L = (2-\beta)C$ with $\beta = 1.05$ or $\beta = 1.10$. For each problem an efficient point with respect to C was generated by solving the linear program $\max\{\lambda Cx : Ax=b, x \geqq 0\}$ with C as defined above and $\lambda = (1,1,1)$ or $(1,1,8)$ when $p = 3$ and $\lambda = (1,1,1,1,1)$ or $(1,1,1,1,6)$ when $p = 5$. The computer program was written in Fortran and was processed on the mini-computer Prime 400. Several branching rules were tested in the algorithm and are not reported here due to space limitations. The use of different values for λ is an attempt to identify a distinct behavior between points that tend to be in the center of the efficient set, i.e. those corresonding to λ's with all components equal, and those that tend to be in the frontier of the efficient set. The computational results are given in Table 5.1. Each row of the table summarizes the results of a sample of 25 problems $P(x°)$. m, n, p, λ, β, μ, and σ denote respectively the number of constraints, number of variables, number of criteria, vector of weights, the number that multiply the randomly generated matrix C to determine $U = [\mu_{ij}]$, the mean c.p.u. time to solve 25 problems $P(x°)$ using the implicit enumeration algorithm (this time does not include the determination of the point $x°$, i.e., the c.p.u. time to solve $\max\{\lambda Cx : Ax=b, x \geqq 0\}$) and the standard deviation of the time to solve each problem $P(x°)$. The last column in Table 5.1 indicates the total number of points that were found efficient after solving the 25 problems $P(x°)$. The computational results show that for $\beta = 1.05$ and

Table 5.1

m	n	p	λ	β	μ 10^{-2} sec	σ 10^{-2} sec	No. of efficient points
10	20	3	(1,1,1)	1,05	.17	.05	20
10	20	3	(1,1,8)	1.05	.42	.20	18
10	20	3	(1,1,1)	1.10	.68	.29	15
10	20	3	(1,1,8)	1.10	.32	.11	10
10	20	5	(1,1,1,1,1)	1.05	1.04	.61	24
10	20	5	(1,1,1,1,6)	1.05	2.55	2.00	25
10	20	5	(1,1,1,1,1)	1.10	4.16	2.01	20
10	20	5	(1,1,1,1,6)	1.10	2.58	1.19	15
10	50	3	(1,1,1)	1.05	2.96	1.81	18
10	50	3	(1,1,8)	1.05	2.77	1.21	18
10	50	3	(1,1,1)	1.10	4.31	2.35	15
10	50	3	(1,1,8)	1.10	2.47	.98	10
25	50	3	(1,1,1)	1.05	19.90	14.22	3
25	50	3	(1,1,8)	1.05	22.67	14.89	3
25	50	3	(1,1,1)	1.10	8.01	5.16	1
25	50	3	(1,1,8)	1.10	5.93	6.70	0

m, n, and p fixed the mean computation time is usually lower for equal weights than for unequal weights, the opposite is true for $\beta = 1.10$. This fact suggests that for small perturbations in the matrix C, i.e., for small values of β, problems $P(x^\circ)$ for x° obtained with equal weights are easier to solve or equivalently, the algorithm needs to explore fewer nodes of the tree than for the case of unequal weights of the type considered. It is also interesting to note that for vectors λ with equal components the number of points found efficient tends to be larger than for the other weights considered. For the ranges of n, m, and p tested, the mean computational time seems to be equally sensitive to individual increases in these parameters. However, the variance is apparently more sensitive to an increase in the number of constraints for $\beta = 1.05$ than for $\beta = 1.10$.

Conclusions and Topics for Further Research

We have considered in this paper the linear multiple objective problem with interval coefficients, denoted by (P). Properties of (P) were presented

together with a problem to test if a feasible extreme point is or not efficient. This problem can be used in conjunction with known algorithms to determine the set of efficient extreme points in (P). A branch and bound algorithm and computational results were given and discussed.

Potential extensions and topics for further research are the determination of "an easy to check" necessary and sufficient condition for the existence of an efficient point in (P) and the consideration of situations where the coefficients of the objective functions are not independent random variables. For the same practical reasons pointed out in the introduction of the paper, it is of interest to develop algorithms to solve the interval coefficient versions of zero-one linear multiple objective programs, the problem of finding efficient points when the set of discrete alternatives is given explicitly [3] and several other discrete programs suggested in [39].

Acknowledgement

The author is grateful to Mrs. Simona Weber for her collaboration in the computational experiments and to Professor Stephen C. Graves for his comments on an earlier version of this paper.

References

1. Bernstein, S. J. and Mellon, W. G., "Multi-Dimensional Considerations in the Evaluations of Urban Policy", in J. L. Cochrane and M. Zeleny (eds.), Multiple Criteria Decision Making, University of South Carolina Press, Columbia, SC, 1973, pp. 530-543.

2. Birkin, S. J. and Ford, J. S., "The Quantity/Quality Dilemma: The Impact of a Zero Defects Program", in J. L. Cochrane and M. Zeleny (eds.), Multiple Criteria Decision Making, University of South Carolina Press, Columbia, SC, 1973, pp. 517-529.

3. Bitran, G. R., "Theory and Algorithms for Linear Multiple Objective Programs with Zero-One Variables", Technical Report No. 150, Operations Research Center, Massachusetts Institute of Technology, May 1978 (to appear in Mathematical Programming).

4. Bitran, G. R. and Magnanti, T. L., "The Structure of Admissible Points with Respect to Cone Dominance", to appear in the Journal of Optimization Theory and Applications.

5. Candea, D., Hax, A. C., and Karmarkar, U., "Economic and Social Evaluation of Capital Investment Decisions - An Application", in A. C. Hax (ed.), Studies in Operations Management, North Holland Pub. Co., Amsterdam, 1978, pp. 205-252.

6. Charnes, A. and Cooper, W. W., "Management Models and Industrial Applications of Linear Programming", Management Science, Vol. 4, No. 1 (1957), pp. 81-87.

7. Cochrane, J. L. and Zeleny, M. (eds.), Multiple Criteria Decision Making, University of South Carolina Press, Columbia, SC, 1973.

8. Cohon, J. L., Multiobjective Programming and Planning, Academic Press, New York, 1978.

9. Collins, D. C., "Applications of Multiple Criteria Evaluation to Decision Aiding", in J. L. Cochrane and M. Zeleny (eds.), Multiple Criteria Decision Making, University of South Carolina Press, Columbia, SC, 1973, pp. 477-504.

10. Dasgupta, P. and Sen, A., Guidelines for Project Evaluation, United Nations, New York, 1972.

11. Ecker, J. G. and Kouada, I. A., "Finding Efficient Points for Linear Multiple Objective Programs", Mathematical Programming, Vol. 8 (1975), pp. 375-377.

12. Evans, J. P. and Steuer, R. E., "A Revised Simplex Method for Linear Multiple Objective Programs", Mathematical Programming, Vol. 5 (1973), pp. 54-72.

13. Farquhar, P. H., "A Survey of Multiattribute Utility Theory and Applications", TIMS Studies in Management Sciences, Vol. 6 (1977), pp. 59-89.

14. Fishburn, P. C., "A Survey of Multiattribute/Multicriterion Evaluation Theories", in S. Zionts (ed.), Multiple Criteria Problem Solving Proceedings, Buffalo, NY (USA), 1977, Springer-Verlag, New York, 1978, pp. 181-224.

15. Gal, T., "An Overview of Recent Results in MCP as Developed in Aachen, Germany", in S. Zionts (ed.), Multiple Criteria Problem Solving Proceedings, Buffalo, NY (USA), 1977, Springer-Verlag, New York, 1978, pp. 225-248.

16. Gal, T., "A General Method for Determining the Set of all Efficient Solutions to a Linear Vector-Maximum Problem", Centre for Operations Research and Econometrics (CORE), Heverlee, Belgium, 1976.

17. Geoffrion, A. M., "Proper Efficiency and the Theory of Vector Maximization", Journal of Mathematical Analysis & Applications, Vol. 22 (1968), pp. 39-54.

18. Fandel, G., "Public Investment Decision Making with Multiple Criteria: An Example of University Planning", in S. Zionts (ed.), Multiple Criteria Problem Solving Proceedings, Buffalo, NY (USA), 1977, Springer-Verlag, New York, 1978, pp. 116-130.

19. Hwang, C. L. and Masud, A. S., Multiple Objective Decision Making Methods Applications: A State of the Art Survey, Springer-Verlag, New York, 1979.

20. Isermann, H., "The Enumeration of the Set of All Efficient Solutions for a Linear Multiple Objective Program", Operations Research Quarterly, Vol. 28 (1977), pp. 711-725.

21. Keefer, D. L., "Applying Multiobjective Decision Analysis to Resource Allocation Planning Problems", in S. Zionts (ed.), Multiple Criteria Problem Solving Proceedings, Buffalo, NY (USA), 1977, Springer-Verlag, New York, 1978, pp. 299-320.

22. Keeney, R. L. and Raiffa, H., Decisions with Multiple Objectives: Preferences and Value Tradeoffs, John Wiley & Sons, New York, 1976.

23. Kornbluth, J. S. H., "A Survey of Goal Programming", OMEGA, Vol. 1, No. 2 (1973), pp. 193-205.

24. Lawrence, K. D. and Lawrence, S. M., "A Multiple Objective Linear Programming Model for Corporate Financial Management", invited paper on the session on Multicriteria Optimization, joint ORSA/TIMS National Meeting, Atlanta, Georgia, November 1977.

25. MacCrimmon, K. R., "An Overview of Multiple Objective Decision Making", In J. L. Cochrane and M. Zeleny (eds.), Multiple Criteria Decision Making, University of South Carolina Press, Columbia, SC, 1973, pp. 18-44.

26. Pasternak, H. and Passy, U., "Bicriterion Functions in Annual Activity Planning", in Proceedings of the Sixth International Conference on Operations Research, Dublin, 1972, pp. 325-341.

27. Philip, J., "Algorithms for the Vector Maximization Problem", Mathematical Programming, Vol. 2 (1972), pp. 207-229.

28. Ross, G. T. and Soland, R. M., "A Multicriteria Approach to the Location of Public Facilities", Rapport Technique No. EP78-R-16, Department de Genie Industriel, Ecole Polytechnique de Montreal, June 1978.

29. Roy, B., "Problems and Methods with Multiple Objective Functions", Mathematical Programming, Vol. 1 (1971), pp. 239-266.

30. Starr, M. and Zeleny, M. (eds.), Multiple Criteria Decision Making, North Holland/TIMS Studies in the Management Sciences, Vol. 6.

31. Starr, M. K., Greenwood, L. H., and Zeleny, M., "MCDM-State and Future of the Arts", in M. K. Starr and M. Zeleny (eds.), Multiple Criteria Decision Making, North Holland, New York, 1977, pp. 5-30.

32. Steuer, R. E., "Multiple Objective Linear Programming with Interval Criterion Weights", Management Science, Vol. 23, No. 3 (1977), pp. 305-316.

33. Steuer, R. E. and Schuler, A. T., "An Interactive Multiple Objective Linear Programming Approach to a Problem in Forest Management", College of Business and Economics, University of Kentucky, Lexinton, Kentucky, 1978.

34. Wallenius, H., Wallenius, J., and Vartia, P., "An Approach to Solving Multiple Criteria Macroeconomic Policy Problems and Applications", Management Science, Vol. 24, No. 10 (1978), pp. 1021-1030.

35. Yu, P. L., "Cone Convexity, Cone Extreme Points, and Nondominated Solutions in Decision Problems with Multipleobjectives", Journal of Optimization Theory and Applications, Vol. 14 (1974), pp. 319-377.

36. Yu, P. L. and Zeleny, M., "The Set of All Nondominated Solutions in Linear Cases and a Multicriteria Simplex Method", Journal of Mathematical Analysis and Applications, Vol. 49 (1975), pp. 430-468.

37. Zionts, S (ed.), Multiple Criteria Problem Solving, Proceedings, Buffalo, NY (USA), 1977, Springer-Verlag, New York, 1978.

38. Zionts, S. and Deshpande, D., "A Time Sharing Computer Programming Application of a Multicriteria Decision Method to Energy Planning - A Progress Report", in S. Zionts (ed.), Multiple Criteria Problem Solving Proceedings, Buffalo, NY (USA), 1977, Springer-Verlag, New York, 1978, pp. 549-560.

39. Zionts, S. and Wallenius, J., "Identifying Efficient Vectors: Some Theory and Computational Results", Working Paper No. 257, SUNY at Buffalo, Buffalo, NY, March 1976.